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# Reduced distribution functions of a microcanonically distributed plasma

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**Abstract.** The exact and limiting reduced distribution functions have been derived for any subsystem of a microcanonically distributed one-component plasma consisting of $N$ electrons surrounded by a homogeneous continuous background of positive charge. In the derivation the Coulomb interaction potential of *all* the electrons of the system was taken into account and retained in the Hamiltonian of the subsystem.

## 1. Introduction

For a closed system, the energy $E$ of which is, by definition, constant, the equilibrium $N$-particle specific distribution function $\rho_N$, called by Gibbs (1902) the microcanonical distribution, can be written in the form (Landau and Lifshitz 1958)

$$\rho_N = C\delta[H(\boldsymbol{p}, \boldsymbol{x}) - E] \tag{1}$$

where $H(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$, written for brevity as $H(\boldsymbol{p}, \boldsymbol{x})$, is the Hamiltonian of the system of $N$ particles expressed in general as a sum of three parts: the kinetic energy of the particles, the potential of external forces acting on the particles including the 'wall potential', and the interaction potential of the particles within the system; $C$ is the normalisation constant

$$1/C = \int \delta[H(\boldsymbol{p}, \boldsymbol{x}) - E]\, \mathrm{d}\boldsymbol{p}_1\, \mathrm{d}\boldsymbol{x}_1 \ldots \mathrm{d}\boldsymbol{p}_N\, \mathrm{d}\boldsymbol{x}_N$$

where the integration with respect to $\mathrm{d}\boldsymbol{x}_1 \ldots \mathrm{d}\boldsymbol{x}_N$ is taken over the domain of space $\Omega$ bounded by the walls of the vessel containing the system and having finite volume $V$.

The distribution (1) is connected with the $s$-particle specific reduced distribution function $\rho_s$ for any subsystem of $s < N$ particles of the microcanonically distributed system through the definition

$$\rho_s = C\int \delta[H(\boldsymbol{p}, \boldsymbol{x}) - E]\, \mathrm{d}\boldsymbol{p}_{s+1}\, \mathrm{d}\boldsymbol{x}_{s+1} \ldots \mathrm{d}\boldsymbol{p}_N\, \mathrm{d}\boldsymbol{x}_N. \tag{2}$$

For a non-closed macroscopic system, the energy of which is a random quantity varying with time, the $N$-particle specific distribution function $\rho_N$ has, in thermal equilibrium,

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the approximate form, called by Gibbs (1902) the canonical distribution:

$$\rho_N = A \exp[-H(\boldsymbol{p}, \boldsymbol{x})/kT] \tag{3}$$

where $T$ is the temperature of the system, $k$ is Boltzmann's constant and $A$ is the normalisation constant,

$$1/A = \int \exp[-H(\boldsymbol{p}, \boldsymbol{x})/kT]\,\mathrm{d}\boldsymbol{p}_1\,\mathrm{d}\boldsymbol{x}_1 \ldots \mathrm{d}\boldsymbol{p}_N\,\mathrm{d}\boldsymbol{x}_N.$$

Again, the $s$-particle specific reduced distribution function $\rho_s$ for any subsystem of $s < N$ particles of the canonically distributed system is defined by

$$\rho_s = A \int \exp[-H(\boldsymbol{p}, \boldsymbol{x})/kT]\,\mathrm{d}\boldsymbol{p}_{s+1}\,\mathrm{d}\boldsymbol{x}_{s+1} \ldots \mathrm{d}\boldsymbol{p}_N\,\mathrm{d}\boldsymbol{x}_N. \tag{4}$$

The microcanonical distribution, written in the form (1) using the Dirac $\delta$ function, represents the mathematical expression for the principle of equal *a priori* probabilities according to which (Gibbs 1902) the specific probability distribution function $\rho_N$ is constant inside the region in $\Gamma$ space ($6N$-dimensional orthogonal Cartesian space) between the two neighbouring energy surfaces $H = E$ and $H = E + \mathrm{d}E$. The canonical distribution (3) introduced by Gibbs (1902) as a postulate for a non-closed system in thermal equilibrium is, in fact, based on the distribution law (1) for the closed system. The many versions presented in various degrees of rigour (Uhlenbeck and Ford 1963, Balescu 1975) of the proof of the canonical distribution (3) are essentially based on the following assumptions.

(a) The system is weakly coupled with a very large 'heat reservoir', i.e. the total Hamiltonian is equal to the sum of the Hamiltonians of the system and the 'heat reservoir', the interaction energy between them, assumed to be small enough for its contribution to the total energy to be negligible.

(b) The total system is in thermal equilibrium, with distribution function the microcanonical distribution.

(c) The system is much smaller than the 'heat reservoir'.

The Hamiltonian of the system in distribution (3) may generally be a non-additive function which includes the interaction potential only of the particles of the system.

The same approach which leads to the canonical distribution (3) for a non-closed system can be followed to derive the distribution function for a small subsystem of $s$ particles of a closed, large system of $N$ particles, the $N - s$ particles now playing the role of the 'heat reservoir' for the remaining $s$ particles. If the subsystem is spatially separated from its complement and interparticle potentials are short-range potentials, then the case is identical with what has been said above for a non-closed system, and under the same assumptions (a), (b) and (c) the subsystem is shown to be canonically distributed, with a Hamiltonian including the interaction potential *only* of the particles of the subsystem. On the other hand, if the subsystem is not spatially separated from its complement and consists of particles identical with the particles of its complement, no matter what the range of interparticle potentials is, it is obvious that application of assumption (a) above implies further that the Hamiltonian of the system has to be additive, i.e. the system has to be considered as ideal. Indeed, it was shown by Khinchin (1949) that a subsystem of $s$ particles of a microcanonically distributed system consisting of an ideal gas of $N$ point particles is canonically distributed if $s \ll N$, with distribution function $\rho_s$ identical to the distribution function obtained for the same subsystem if the whole system of $N$ particles was considered to be canonically distributed.

Therefore a question arises: if we consider a microcanonically distributed system of $N$ particles, what is the distribution function for any subsystem of $s < N$ particles not spatially separated from the $N-s$ particles, if a non-additive Hamiltonian is to be retained for the system, i.e. if the system is to be considered as non-ideal? In this work we attempted to answer rigorously the above question for a closed, neutral system consisting of charged particles with pure Coulomb interactions.

## 2. Reduced distribution functions of a microcanonically distributed plasma

Let us consider a system consisting of electrically charged particles (a plasma) with Coulomb interaction potentials which, although weak, are long-range potentials. In a real plasma there must be at least two components present, say, electrons and positive ions, so that the total charge of the system is zero. In the following analysis, however, we shall consider a simplified model of a plasma: a one-component system, say, an electron gas, in the presence of a homogeneous continuous background of positive charge which neutralises the overall charge but has no dynamic role and is not retained in the Hamiltonian of the system (Balescu 1975). We assume that the system considered is confined by hard reflective walls to a domain $\Omega$ of physical space and is not subject to any external forces other than the 'wall potential' $U_W(\boldsymbol{x}_j)$ acting on the $j$th electron ($j = 1, 2, \ldots, N$) and assumed to be $U_W(\boldsymbol{x}_j) = 0$ inside the vessel and $U_W(\boldsymbol{x}_j) = \infty$ outside the vessel. Also, the system under consideration is closed, in thermal equilibrium and characterised by a constant energy $E$. The Hamiltonian of the system including the interaction potential energy $U$ of all the electrons is

$$H(\boldsymbol{p}, \boldsymbol{x}) = \sum_{j=1}^{N} \frac{\boldsymbol{p}_j^2}{2m} + \sum_{j<i=1}^{N} \frac{e^2}{\|\boldsymbol{x}_j - \boldsymbol{x}_i\|} = E, \qquad \boldsymbol{x}_j \in \Omega | j = 1, 2, \ldots, N, \tag{5}$$

where $\Sigma_{j<i=1}^{N} e^2/\|\boldsymbol{x}_j - \boldsymbol{x}_i\| = U$, and $e$ and $m$ are the charge and mass, respectively, of each electron.

We introduce now the following dimensionless momentum and position variables $\boldsymbol{p}'_j$ and $\boldsymbol{x}'_j$:

$$\boldsymbol{p}'_j = \boldsymbol{p}_j (N/2mE)^{1/2}, \qquad \boldsymbol{x}'_j = \boldsymbol{x}_j (E/Ne^2). \tag{6}$$

The presence of the number of electrons $N$ in the transformations (6) ensures that the new variables $\boldsymbol{p}'_j$ and $\boldsymbol{x}'_j$ remain finite at the thermodynamic limit: $N \to \infty$, $E \to \infty$. In terms of the dimensionless variables $\boldsymbol{p}'_j$ and $\boldsymbol{x}'_j$ the Hamiltonian of the system, equation (5), is written as

$$H'(\boldsymbol{p}', \boldsymbol{x}') = \sum_{j=1}^{N} \frac{\boldsymbol{p}_j'^2}{N} + \sum_{j<i=1}^{N} \frac{1}{N\|\boldsymbol{x}'_j - \boldsymbol{x}'_i\|} = 1, \qquad \boldsymbol{x}'_j \in \Omega' | j = 1, 2, \ldots, N. \tag{7}$$

The principle of equal *a priori* probabilities (Gibbs 1902) applied in the energy shell between the two neighbouring energy surfaces $H' = 1$ and $H' = 1 + \mathrm{d}E/E$ in the transformed space, $\Gamma'$ space, gives for the specific probability distribution function $\rho'_N = \rho'_N(\boldsymbol{p}', \boldsymbol{x}')$ of the system of $N$ electrons the microcanonical distribution

$$\rho'_N = C'\delta[H'(\boldsymbol{p}', \boldsymbol{x}') - 1] \tag{8}$$

which is in turn connected with the $s$-particle reduced distribution function $\rho'_s$, for any subsystem of $s<N$ electrons, through the formula

$$\rho'_s = C' \int \delta[H'(\boldsymbol{p}', \boldsymbol{x}')-1]\, \mathrm{d}\boldsymbol{p}'_{s+1}\, \mathrm{d}\boldsymbol{x}'_{s+1} \ldots \mathrm{d}\boldsymbol{p}'_N\, \mathrm{d}\boldsymbol{x}'_N \tag{9}$$

where $C'$ is the normalisation constant:

$$\frac{1}{C'} = \int \delta[H'(\boldsymbol{p}', \boldsymbol{x}')-1]\, \mathrm{d}\boldsymbol{p}'_1\, \mathrm{d}\boldsymbol{x}'_1 \ldots \mathrm{d}\boldsymbol{p}'_N\, \mathrm{d}\boldsymbol{x}'_N. \tag{10}$$

Each differential volume $\mathrm{d}\boldsymbol{p}'_1\, \mathrm{d}\boldsymbol{x}'_1 \ldots \mathrm{d}\boldsymbol{p}'_N\, \mathrm{d}\boldsymbol{x}'_N$ in $\Gamma'$ space can be written as $\mathrm{d}\boldsymbol{p}'_1\, \mathrm{d}\boldsymbol{x}'_1 \ldots \mathrm{d}\boldsymbol{p}'_N\, \mathrm{d}\boldsymbol{x}'_N = \mathrm{d}n\, \mathrm{d}S$ where $\mathrm{d}n$ is the normal distance between the energy surfaces $H'(\boldsymbol{p}', \boldsymbol{x}') = H'$ and $H'(\boldsymbol{p}', \boldsymbol{x}') = H' + \mathrm{d}H'$, and $\mathrm{d}S$ is a differential surface element of $H'(\boldsymbol{p}', \boldsymbol{x}') = H'$. But $\mathrm{d}n = \mathrm{d}H'/\|\mathrm{grad}\, H'\|$ and thus:

$$\mathrm{d}\boldsymbol{p}'_1\, \mathrm{d}\boldsymbol{x}'_1 \ldots \mathrm{d}\boldsymbol{p}'_N\, \mathrm{d}\boldsymbol{x}'_N = \frac{\mathrm{d}H'\, \mathrm{d}S}{\|\mathrm{grad}\, H'\|}. \tag{11}$$

From equation (7) we can calculate grad $H'$ and hence $\|\mathrm{grad}\, H'\|$ as:

$$\|\mathrm{grad}\, H'\| = \left\{ \sum_{j=1}^{N} \left[ \left( \sum_{\substack{i=1 \\ i\neq j}}^{N} \frac{\boldsymbol{x}'_j - \boldsymbol{x}'_i}{N\|\boldsymbol{x}'_j - \boldsymbol{x}'_i\|^3} \right)^2 + 4\left(\frac{\boldsymbol{p}'_j}{N}\right)^2 \right] \right\}^{1/2}. \tag{12}$$

Using equation (11) we can write for the reduced distribution function (9):

$$\rho'_s\, \mathrm{d}\boldsymbol{p}'_1\, \mathrm{d}\boldsymbol{x}'_1 \ldots \mathrm{d}\boldsymbol{p}'_s\, \mathrm{d}\boldsymbol{x}'_s = C' \int \delta[H'(\boldsymbol{\zeta}', \boldsymbol{\eta}')-1]\mathscr{D}\, \mathrm{d}\boldsymbol{\zeta}'_1\, \mathrm{d}\boldsymbol{\eta}'_1 \ldots \mathrm{d}\boldsymbol{\zeta}'_N\, \mathrm{d}\boldsymbol{\eta}'_N$$

$$= C' \int_{H'=0}^{H'=+\infty} \mathrm{d}H' \delta(H'-1) \int_{S(H')} \frac{\mathscr{D}}{\|\mathrm{grad}\, H'\|}\, \mathrm{d}S \tag{13}$$

where

$$\mathscr{D} = \mathrm{d}\boldsymbol{p}'_1\, \mathrm{d}\boldsymbol{x}'_1 \ldots \mathrm{d}\boldsymbol{p}'_s\, \mathrm{d}\boldsymbol{x}'_s \prod_{j=1}^{s} \delta(\boldsymbol{\zeta}'_j - \boldsymbol{p}'_j) \prod_{j=1}^{s} \delta(\boldsymbol{\eta}'_j - \boldsymbol{x}'_j)$$

$= 1$ when $\boldsymbol{\zeta}'_j \in \mathrm{d}\boldsymbol{p}'_j$ and $\boldsymbol{\eta}'_j \in \mathrm{d}\boldsymbol{x}'_j$ for all $j = 1, 2, \ldots, s$

$= 0$ otherwise

and $S(H')$ is the surface of $H'(\boldsymbol{\zeta}', \boldsymbol{\eta}') = H'$.

Performing the integration with respect to $\mathrm{d}H'$ in equation (13) we further obtain:

$$\rho'_s\, \mathrm{d}\boldsymbol{p}'_1\, \mathrm{d}\boldsymbol{x}'_1 \ldots \mathrm{d}\boldsymbol{p}'_s\, \mathrm{d}\boldsymbol{x}'_s = C' \int_Z \frac{\mathrm{d}S}{\|\mathrm{grad}\, H'\|}. \tag{14}$$

The integration in equation (14) is taken over a differential region $Z$ of the surface $H'(\boldsymbol{\zeta}', \boldsymbol{\eta}') = 1$, confined between the $6(N-s)$-dimensional planes $\boldsymbol{\zeta}'_j = \boldsymbol{p}'_j$; $\boldsymbol{\eta}'_j = \boldsymbol{x}'_j$; $j = 1, 2, \ldots, s$ and $\boldsymbol{\zeta}'_j = \boldsymbol{p}'_j + \mathrm{d}\boldsymbol{p}'_j$; $\boldsymbol{\eta}'_j = \boldsymbol{x}'_j + \mathrm{d}\boldsymbol{x}'_j$; $j = 1, 2, \ldots, s$.

We can now write equation (7) in the form:

$$\sum_{j=s+1}^{N} \boldsymbol{p}'^2_j = N\left(1 - \sum_{j<i=1}^{N} \frac{1}{N\|\boldsymbol{x}'_j - \boldsymbol{x}'_i\|} - \sum_{j=1}^{s} \frac{\boldsymbol{p}'^2_j}{N}\right) \geq 0. \tag{7a}$$

Equation ($7a$) shows that the surface $H'(\boldsymbol{p}', \boldsymbol{x}') = 1$ can be considered as a surface of revolution whose 'axis' of symmetry is the $3(N+s)$-dimensional space

$(\boldsymbol{p}'_1, \ldots, \boldsymbol{p}'_s, \boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_N)$. For each specific point in this space a $3(N-s)$-dimensional sphere immersed in the space $(\boldsymbol{p}'_{s+1}, \ldots, \boldsymbol{p}'_N)$ is obtained:

$$\sum_{j=s+1}^{N} \boldsymbol{p}'^2_j = R_s^2 \tag{7b}$$

with radius $|R_s|$:

$$R_s = \pm\left(1 - \sum_{j<i=1}^{N} \frac{1}{N\|\boldsymbol{x}'_j - \boldsymbol{x}'_i\|} - \sum_{j=1}^{s} \frac{\boldsymbol{p}'^2_j}{N}\right)^{1/2} \sqrt{N}. \tag{15}$$

The fact that $\|\text{grad}\, H'\|$ in equation (12), looked upon as a function of $\boldsymbol{p}'_{s+1}, \ldots, \boldsymbol{p}'_N$, depends only on $\boldsymbol{p}'^2_{s+1} + \ldots + \boldsymbol{p}'^2_N$ suggests that we can choose as a surface element $\mathrm{d}S$ of $H'(\boldsymbol{p}', \boldsymbol{x}') = 1$ the element

$$\mathrm{d}S = \mathrm{d}l \int_{F_s} \mathrm{d}F_s = 2\frac{\pi^{3(N-s)/2}}{\Gamma[3(N-s)/2]} |R_s|^{3(N-s)-1}\, \mathrm{d}l \tag{16}$$

where the integration in equation (16) was taken over the surface $F_s$ of the sphere ($7b$), and $\mathrm{d}l$ is given according to the first fundamental form of the surface $R_s = R_s(\boldsymbol{p}'_1, \ldots, \boldsymbol{p}'_s, \boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_N)$, by

$$\mathrm{d}l = \left[1 + \sum_{j=1}^{s} \left(\frac{\partial R_s}{\partial \boldsymbol{p}'_j}\right)^2 + \sum_{j=1}^{N} \left(\frac{\partial R_s}{\partial \boldsymbol{x}'_j}\right)^2\right]^{1/2} \mathrm{d}\boldsymbol{p}'_1 \ldots \mathrm{d}\boldsymbol{p}'_s\, \mathrm{d}\boldsymbol{x}'_1 \ldots \mathrm{d}\boldsymbol{x}'_N. \tag{17}$$

We can now express $\|\text{grad}\, H'\|$ given by equation (12) in terms of $R_s$ and its derivatives. Taking partial derivatives of $R_s$ from equation (15) we have:

$$\frac{\boldsymbol{p}'_j}{N} = -\frac{R_s}{N} \frac{\partial R_s}{\partial \boldsymbol{p}'_j} \tag{18}$$

and

$$\sum_{\substack{i=1 \\ i \neq 1}}^{N} \frac{\boldsymbol{x}'_j - \boldsymbol{x}'_i}{N\|\boldsymbol{x}'_j - \boldsymbol{x}'_i\|^3} = 2\frac{R_s}{N} \frac{\partial R_s}{\partial \boldsymbol{x}'_j}. \tag{19}$$

Substituting equations (18) and (19) into equation (12) and taking into account equation ($7b$) we obtain

$$\|\text{grad}\, H'\| = 2\frac{|R_s|}{N}\left[1 + \sum_{j=1}^{s} \left(\frac{\partial R_s}{\partial \boldsymbol{p}'_j}\right)^2 + \sum_{j=1}^{N} \left(\frac{\partial R_s}{\partial \boldsymbol{x}'_j}\right)^2\right]^{1/2}. \tag{20}$$

Substitution now of equations (16) and (20), with $\mathrm{d}l$ given by equation (17), into equation (14) gives:

$$\rho'_s = C' \frac{\pi^{3(N-s)/2}}{\Gamma[3(N-s)/2]} N \int |R_s|^{3(N-s)-2}\, \mathrm{d}\boldsymbol{x}'_{s+1} \ldots \mathrm{d}\boldsymbol{x}'_N. \tag{21}$$

The normalisation constant $C'$ can be calculated from equation (10) or equivalently from

$$\frac{1}{C'} = \frac{1}{C'} \int \rho'_s\, \mathrm{d}\boldsymbol{p}'_1\, \mathrm{d}\boldsymbol{x}'_1 \ldots \mathrm{d}\boldsymbol{p}'_s\, \mathrm{d}\boldsymbol{x}'_s$$

$$= \frac{\pi^{3(N-s)/2}}{\Gamma[3(N-s)/2]} N \int \mathrm{d}\boldsymbol{x}'_1 \ldots \mathrm{d}\boldsymbol{x}'_N \int |R_s|^{3(N-s)-2}\, \mathrm{d}\boldsymbol{p}'_1 \ldots \mathrm{d}\boldsymbol{p}'_s. \tag{22}$$

The integral with respect to $\mathrm{d}\boldsymbol{p}'_1 \ldots \mathrm{d}\boldsymbol{p}'_s$ in equation (22) can be calculated using multidimensional polar coordinates in the space $(\boldsymbol{p}'_1, \ldots, \boldsymbol{p}'_s)$. Equation (22) is then written as:

$$\frac{1}{C'} = \frac{\pi^{3(N-s)/2}}{\Gamma[3(N-s)/2]} N \int \mathrm{d}\boldsymbol{x}'_1 \ldots \mathrm{d}\boldsymbol{x}'_N \int (R_0^2 - r^2)^{3(N-s)/2-1} r^{3s-1}\, \mathrm{d}r\, \mathrm{d}\omega \tag{23}$$

where

$$R_0 = \pm\left(1 - \sum_{j<i=1}^{N} \frac{1}{N\|\boldsymbol{x}'_j - \boldsymbol{x}'_i\|}\right)^{1/2} \sqrt{N}, \qquad r = \left(\sum_{j=1}^{s} \boldsymbol{p}'^2_j\right)^{1/2},$$

and performing the integration in equation (23) with respect to $\mathrm{d}r\, \mathrm{d}\omega$ we obtain for $C'$:

$$C' = \frac{\Gamma(3N/2)}{\pi^{3N/2} N} \frac{1}{\int |R_0|^{3N-2}\, \mathrm{d}\boldsymbol{x}'_1 \ldots \mathrm{d}\boldsymbol{x}'_N}. \tag{24}$$

Substitution of $C'$ as given by equation (24) into the distribution (21) gives:

$$\rho'_s = \left(\frac{1}{N\pi}\right)^{3s/2} \frac{\Gamma(3N/2)}{\Gamma[3(N-s)/2]} \frac{\int |R_s|^{3(N-s)-2}\, \mathrm{d}\boldsymbol{x}'_{s+1} \ldots \mathrm{d}\boldsymbol{x}'_N}{\int |R_0|^{3N-2}\, \mathrm{d}\boldsymbol{x}'_1 \ldots \mathrm{d}\boldsymbol{x}'_N}. \tag{25}$$

Transforming distribution (25) into the dimensional variables $\boldsymbol{p}_j$ and $\boldsymbol{x}_j$ we finally obtain for the reduced distribution function $\rho_s$ the result:

$$\rho_s = \frac{\Gamma(3N/2)}{\Gamma[3(N-s)/2]} \left(\frac{1}{2\pi m E}\right)^{3s/2} \frac{\int (1 - H_s/E)^{3(N-s)/2-1}\, \mathrm{d}\boldsymbol{x}_{s+1} \ldots \mathrm{d}\boldsymbol{x}_N}{\int (1 - U/E)^{3N/2-1}\, \mathrm{d}\boldsymbol{x}_1 \ldots \mathrm{d}\boldsymbol{x}_N} \tag{26}$$

where

$$H_s = \sum_{j=1}^{s} \boldsymbol{p}_j^2/2m + U, \qquad U = \sum_{j<i=1}^{N} \frac{e^2}{\|\boldsymbol{x}_j - \boldsymbol{x}_i\|}.$$

Distribution (26) is the exact $s$-particle specific reduced distribution function for a one-component plasma, valid for every $N > 1$ and $s < N$.

If we assume that $N \gg 1$, and use the relation $E = N\langle E\rangle = 3NkT/2$, then distribution (26) is approximated for every $s < N$, $N \gg 1$, by:

$$\rho_s = \frac{\Gamma(3N/2)}{\Gamma[3(N-s)/2]} \left(\frac{1}{3\pi m N k T}\right)^{3s/2} \exp\left[-(1 - s/N) \sum_{j=1}^{s} \boldsymbol{p}_j^2/2mkT\right]$$
$$\times \frac{\int \exp[-(1 - s/N)U/kT]\, \mathrm{d}\boldsymbol{x}_{s+1} \ldots \mathrm{d}\boldsymbol{x}_N}{\int \exp(-U/kT)\, \mathrm{d}\boldsymbol{x}_1 \ldots \mathrm{d}\boldsymbol{x}_N}. \tag{27}$$

If we further assume that $s \ll N$, then distribution (27) reduces to

$$\rho_s = \left(\frac{1}{2\pi m k T}\right)^{3s/2} \exp\left(-\sum_{j=1}^{s} \boldsymbol{p}_j^2/2mkT\right) \frac{\int \exp(-U/kT)\, \mathrm{d}\boldsymbol{x}_{s+1} \ldots \mathrm{d}\boldsymbol{x}_N}{\int \exp(-U/kT)\, \mathrm{d}\boldsymbol{x}_1 \ldots \mathrm{d}\boldsymbol{x}_N}. \tag{28}$$

It can easily be seen that the resulting reduced distribution function (28) for a subsystem of $s \ll N$ electrons of a macroscopic, microcanonically distributed one-component plasma is identical with the reduced distribution function which would be obtained for the same subsystem if the plasma was considered to be canonically distributed.

It would be interesting to note that the same way of approach used above to derive the reduced distribution functions can be followed to derive the $N$-particle position distribution function $D(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) = \int \rho_N \, \mathrm{d}\boldsymbol{p}_1 \ldots \mathrm{d}\boldsymbol{p}_N$ of a microcanonically distributed one-component plasma. Working in the same transformed space, $\Gamma'$ space, but writing equation (7) in the form

$$\sum_{j=1}^{N} \boldsymbol{p}_j'^2 = N\left(1 - \sum_{j<i=1}^{N} \frac{1}{N\|\boldsymbol{x}_j' - \boldsymbol{x}_i'\|}\right) \tag{7c}$$

we now have for each specific point of the $3N$-dimensional space $(\boldsymbol{x}_1', \ldots, \boldsymbol{x}_N')$ a $3N$-dimensional sphere $\sum_{j=1}^{N} \boldsymbol{p}_j'^2 = R_0^2$ of radius $|R_0|$:

$$R_0 = \pm\left(1 - \sum_{j<i=1}^{N} \frac{1}{N\|\boldsymbol{x}_j' - \boldsymbol{x}_i'\|}\right)^{1/2} \sqrt{N} \tag{29}$$

and if we carry on as previously we obtain for the $N$-particle position distribution function, with $N \gg 1$, the result:

$$D = \frac{\exp(-U/kT)}{\int \exp(-U/kT) \, \mathrm{d}T) \, \mathrm{d}\boldsymbol{x}_1 \ldots \mathrm{d}\boldsymbol{x}_N}. \tag{30}$$

It can easily be seen that the resulting $N$-particle position distribution function (30) derived for a microcanonically distributed plasma is identical with the $N$-particle position distribution function which would have been obtained had the plasma been considered as canonically distributed.

## 3. Conclusions

In the present work we considered a microcanonically distributed one-component plasma consisting of $N > 1$ electrons surrounded by a homogeneous continuous background of positive charge, and derived the exact reduced distribution function given by equation (26) for any subsystem of $s < N$ electrons when the Coulomb interaction potential of *all* the electrons is taken into account and retained in the Hamiltonian of the subsystem (and the system). The resulting distribution (26) reduces for $N \gg 1$, $s \ll N$ to distribution (28) which is, in fact, identical to the reduced distribution function that would be obtained for the same subsystem if the given plasma was considered to be canonically distributed. Also, the $N$-particle position distribution function is derived for the given microcanonically distributed plasma following essentially the same way of approach and is given for $N \gg 1$ by equation (30). Distribution (30) is identical with the $N$-particle position distribution function which would have been obtained had the plasma been considered as canonically distributed.

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